

## REGULAR PAIRINGS OF FUNCTORS AND WEAK (CO)MONADS

ROBERT WISBAUER

ABSTRACT. For functors  $L : \mathbb{A} \rightarrow \mathbb{B}$  and  $R : \mathbb{B} \rightarrow \mathbb{A}$  between any categories  $\mathbb{A}$  and  $\mathbb{B}$ , a *pairing* is defined by maps, natural in  $A \in \mathbb{A}$  and  $B \in \mathbb{B}$ ,

$$\mathrm{Mor}_{\mathbb{B}}(L(A), B) \xrightleftharpoons[\beta]{\alpha} \mathrm{Mor}_{\mathbb{A}}(A, R(B)) .$$

$(L, R)$  is an *adjoint pair* provided  $\alpha$  (or  $\beta$ ) is a bijection. In this case the composition  $RL$  defines a monad on the category  $\mathbb{A}$ ,  $LR$  defines a comonad on the category  $\mathbb{B}$ , and there is a well-known correspondence between monads (or comonads) and adjoint pairs of functors.

For various applications it was observed that the conditions for a unit of a monad was too restrictive and weakening it still allowed for a useful generalised notion of a monad. This led to the introduction of *weak monads* and *weak comonads* and the definitions needed were made without referring to this kind of adjunction. The motivation for the present paper is to show that these notions can be naturally derived from pairings of functors  $(L, R, \alpha, \beta)$  with  $\alpha = \alpha \cdot \beta \cdot \alpha$  and  $\beta = \beta \cdot \alpha \cdot \beta$ . Following closely the constructions known for monads (and unital modules) and comonads (and counital comodules), we show that any weak (co)monad on  $\mathbb{A}$  gives rise to a regular pairing between  $\mathbb{A}$  and the category of *compatible (co)modules*.

MSC: 18A40, 18C20, 16T15.

*Keywords:* pairing of functors; adjoint functors; weak (co)monads;  $r$ -unital monads;  $r$ -counital comonads; lifting of functors; distributive laws.

## CONTENTS

1. Introduction	1
2. Pairings of functors	3
3. Monads and modules	5
4. Comonads and comodules	8
5. Entwining monads and comonads	11
6. Lifting of endofunctors to modules and comodules	13
7. Mixed entwining and liftings	15
References	18

## 1. INTRODUCTION

Similar to the unit of an algebra, the existence of a unit of a monad is essential for (most of) the interesting properties of the related structures. Yet, there are numerous applications for which the request for a unit of a monad is too restrictive. Dropping the unit completely makes the theory fairly poor and the question was how to weaken the conditions on a unit such that still an effective theory can be developed. The interest in these questions was revived, for example, by the study of *weak Hopf algebras* by G. Böhm et al. in [6] and *weak entwining structures* by S. Caenepeel et al. in [9] (see also [1], [8]). To handle this situation the theory of weak monads and comonads was developed and we refer to [3] for a recent account on this theory.

On any category, monads are induced by a pair of adjoint functors and, on the other hand, any monad  $(F, \mu, \eta)$  induces an adjoint pair of functors, the free functor  $\phi_F : \mathbb{A} \rightarrow \mathbb{A}_F$

and the forgetful functor  $U_F : \mathbb{A}_F \rightarrow \mathbb{A}$ , where  $\mathbb{A}_F$  denotes the category of unital  $F$ -modules. This is all shown in Eilenberg-Moore [10].

In this correspondence the unitality of the monad is substantial and the purpose of the present paper is to exhibit a similar relationship between weak (co)monads and generalised forms of adjunctions. To this end, for functors  $L : \mathbb{A} \rightarrow \mathbb{B}$  and  $R : \mathbb{B} \rightarrow \mathbb{A}$  between categories  $\mathbb{A}$  and  $\mathbb{B}$ , we consider maps

$$\mathrm{Mor}_{\mathbb{B}}(L(A), B) \xrightleftharpoons[\beta]{\alpha} \mathrm{Mor}_{\mathbb{A}}(A, R(B)) ,$$

required to be natural in  $A \in \mathbb{A}$  and  $B \in \mathbb{B}$ . We call this a *pairing of functors*, or a *full pairing* if we want to stress that we have maps in both directions. Such a pairing is said to be *regular* provided  $\alpha$  and  $\beta$  are regular maps, more precisely,

$$\alpha = \alpha \cdot \beta \cdot \alpha \quad \text{and} \quad \beta = \beta \cdot \alpha \cdot \beta .$$

In Section 2, regular pairings of functors are defined and some of their general properties are described.

Motivated by substructures showing up in pairings of functors, in Section 3.1, *q-unital monads*  $(F, \mu, \eta)$  on  $\mathbb{A}$  are defined as endofunctors  $F : \mathbb{A} \rightarrow \mathbb{A}$  with natural transformations  $\mu : FF \rightarrow F$  and  $\eta : I_{\mathbb{A}} \rightarrow F$  (*quasi-unit*) and the sole condition that  $\mu$  is associative. (*Non-unital*) *F-modules* are defined by morphisms  $\varrho : F(A) \rightarrow A$  satisfying  $\varrho \circ \mu = \varrho \circ F\varrho$ , and the category of all  $F$ -modules is denoted by  $\underline{\mathbb{A}}_F$ . For these data the free and forgetful functors,

$$\phi_F : \mathbb{A} \rightarrow \underline{\mathbb{A}}_F \quad \text{and} \quad U_F : \underline{\mathbb{A}}_F \rightarrow \mathbb{A} .$$

give rise to a full pairing. From this we define *regularity* of  $\eta$  and *compatibility* for the  $F$ -modules. The  $q$ -unital monad  $(F, \mu, \eta)$  is said to be *r-unital* (short for *regular-unital*) provided  $\eta$  is regular and  $\mu$  is compatible as an  $F$ -module. Now the free functor  $\phi_F : \mathbb{A} \rightarrow \underline{\mathbb{A}}_F$  with the forgetful functor  $U_F : \underline{\mathbb{A}}_F \rightarrow \mathbb{A}$  form a regular pairing, where  $\underline{\mathbb{A}}_F$  denotes the (sub)category of compatible  $F$ -modules.

The dual notions for (non-counital) comonads are outlined in Section 4 and at the end of the section the comparison functors for a regular pairing  $(L, R, \alpha, \beta)$  are considered (see 4.10).

In Section 5 we study the lifting of functors between categories to the corresponding categories of compatible modules or compatible comodules, respectively. This is described by generalising Beck's *distributive laws* (see [2]), also called *entwinings*, and it turns out that most of the diagrams are the same as for the lifting to unital modules (e.g. [22]) but to compensate the missing unitality extra conditions are imposed on the entwining natural transformation (e.g. Proposition 5.2). In this context we obtain a generalisation of Applegate's lifting theorem for (co)monads to weak (co)monads (Theorem 5.4, 5.8).

Lifting an endofunctor  $T$  of  $\mathbb{A}$  to an endofunctor  $\overline{T}$  of  $\underline{\mathbb{A}}_F$  leads to the question when  $\overline{T}$  is a weak monad ( $TF$  allows for the structure of a weak monad) and in Section 6 we provide conditions to make this happen.

The final Section 7 is concerned with weak monads  $(F, \mu, \eta)$  and weak comonads  $(G, \delta, \varepsilon)$  on any category  $\mathbb{A}$  and the interplay between the respective lifting properties. Hereby properties of the lifting  $\overline{G}$  to  $\underline{\mathbb{A}}_F$  and the lifting  $\widehat{F}$  to  $\underline{\mathbb{A}}^G$  are investigated (see Theorems 7.9 and 7.10) which generalise observations known for weak bi-algebras (and weak Hopf algebras).

In our setting, notions like *pre-units*, *pre-monads*, *weak monads*, *demi-monads*, *pre-A-corings*, *weak corings*, *weak Hopf algebras* from the literature (e.g. [1], [3], [7], [4], [21]) find their natural environment.

In the framework of 2-categories weak structures are investigated by Böhm et al. in [3], [4] and an extensive list of examples of weak structures is given there.

## 2. PAIRINGS OF FUNCTORS

Throughout  $\mathbb{A}$  and  $\mathbb{B}$  will denote arbitrary categories. By  $I_A$ ,  $A$  or just by  $I$ , we denote the identity morphism of an object  $A \in \mathbb{A}$ ,  $I_F$  or  $F$  stands for the identity natural transformation on the functor  $F$ , and  $I_{\mathbb{A}}$  means the identity functor of a category  $\mathbb{A}$ . We write  $F_{-, -}$  for the natural transformation of bifunctors determined by the maps  $F_{A, A'} : \text{Mor}_{\mathbb{A}}(A, A') \rightarrow \text{Mor}_{\mathbb{B}}(F(A), F(A'))$  for  $A, A' \in \mathbb{A}$ .

Before considering regularity for natural transformations we recall basic properties of

**2.1. Regular morphisms.** Let  $A, A'$  be any objects in a category  $\mathbb{A}$ . Then a morphism  $f : A \rightarrow A'$  is called *regular* provided there is a morphism  $g : A' \rightarrow A$  with  $fgf = f$ . Clearly, in this case  $gf : A \rightarrow A$  and  $fg : A' \rightarrow A'$  are idempotent endomorphisms.

Such a morphism  $g$  is not necessarily unique. In particular, for  $fgf$  we also have  $f(gfg)f = fgf = f$ , and the identity  $(gfg)f(gfg) = gfg$  shows that  $gfg$  is again a regular morphism.

If idempotents split in  $\mathbb{A}$ , then every idempotent morphism  $e : A \rightarrow A$  determines a subobject of  $A$ , we denote it by  $eA$ .

If  $f$  is regular with  $fgf = f$ , then the restriction of  $fg$  is the identity morphism on  $fgA'$  and  $gf$  is the identity on  $gfA$ .

Examples for regular morphisms are retractions, coretractions, and isomorphisms. For modules  $M, N$  over any ring, a morphism  $f : M \rightarrow N$  is regular if and only if the image and the kernel of  $f$  are direct summands in  $N$  and  $M$ , respectively.

This notion of regularity is derived from von Neumann regularity of rings. For modules (and in preadditive categories) it was considered by Nicholson, Kasch, Mader and others (see [14]). We use the terminology also for natural transformations and functors with obvious interpretations.

**2.2. Pairing of functors.** (e.g. [19, 2.1]) Let  $L : \mathbb{A} \rightarrow \mathbb{B}$  and  $R : \mathbb{B} \rightarrow \mathbb{A}$  be covariant functors. Assume there are morphisms, natural in  $A \in \mathbb{A}$  and  $B \in \mathbb{B}$ ,

$$\begin{aligned} \alpha : \text{Mor}_{\mathbb{B}}(L(A), B) &\rightarrow \text{Mor}_{\mathbb{A}}(A, R(B)), \\ \beta : \text{Mor}_{\mathbb{A}}(A, R(B)) &\rightarrow \text{Mor}_{\mathbb{B}}(L(A), B). \end{aligned}$$

These maps correspond to natural transformations between functors  $\mathbb{A}^{op} \times \mathbb{B} \rightarrow \text{Set}$ . The quadruple  $(L, R, \alpha, \beta)$  is called a *(full) pairing (of functors)*.

Given such a pairing, the morphisms, for  $A \in \mathbb{A}$ ,  $B \in \mathbb{B}$ ,

$$\eta_A := \alpha_{A, L(A)}(I) : A \rightarrow RL(A) \quad \text{and} \quad \varepsilon_B := \beta_{R(B), B}(I) : LR(B) \rightarrow B$$

correspond to natural transformations

$$\eta : I_{\mathbb{A}} \rightarrow RL, \quad \varepsilon : LR \rightarrow I_{\mathbb{B}},$$

which we call *quasi-unit* and *quasi-counit* of  $(L, R, \alpha, \beta)$ , respectively.

From these the transformations  $\alpha$  and  $\beta$  are obtained by

$$\begin{aligned} \alpha_{A, B} : L(A) &\xrightarrow{f} B \mapsto A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B), \\ \beta_{A, B} : A &\xrightarrow{g} R(B) \mapsto L(A) \xrightarrow{L(g)} LR(B) \xrightarrow{\varepsilon_B} B. \end{aligned}$$

Thus the pairing  $(L, R, \alpha, \beta)$  is also described by the quadruple  $(L, R, \eta, \varepsilon)$ .

Naturality of  $\varepsilon$  and  $\eta$  induces an associative product and a quasi-unit for the endofunctor  $RL : \mathbb{A} \rightarrow \mathbb{A}$ ,

$$R\varepsilon L : RLRL \rightarrow RL, \quad \eta : I_{\mathbb{A}} \rightarrow RL,$$

and a coassociative coproduct and a quasi-counit for the endofunctor  $LR : \mathbb{B} \rightarrow \mathbb{B}$ ,

$$L\eta R : LR \rightarrow LRLR, \quad \varepsilon : LR \rightarrow I_{\mathbb{B}}.$$

By the Yoneda Lemma we can describe compositions of  $\alpha$  and  $\beta$  by the images of the identity transformations of the respective functors.

**2.3. Composing  $\alpha$  and  $\beta$ .** Let  $(L, R, \alpha, \beta)$  be a pairing with quasi-unit  $\eta$  and quasi-counit  $\varepsilon$ . The descriptions of  $\alpha$  and  $\beta$  in 2.2 yield, for the identity transformations  $I_L : L \rightarrow L$ ,  $I_R : R \rightarrow R$ ,

$$\begin{aligned} \alpha(I_L) &= I_{\mathbb{A}} \xrightarrow{\eta} RL, \\ \beta \cdot \alpha(I_L) &= L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L, \\ \alpha \cdot \beta \cdot \alpha(I_L) &= I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL, \\ \beta(I_R) &= LR \xrightarrow{\varepsilon} I_{\mathbb{B}}, \\ \alpha \cdot \beta(I_R) &= R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R, \\ \beta \cdot \alpha \cdot \beta(I_R) &= LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR \xrightarrow{\varepsilon} I_{\mathbb{B}}. \end{aligned}$$

The following morphisms will play a special role in what follows.

**2.4. Natural endomorphisms.** With the notions from 2.2, we define the natural transformations

$$\begin{aligned} \vartheta &:= R(\beta\alpha(I_L)) : RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL, \\ \underline{\vartheta} &:= \alpha\beta(R(I_L)) : RL \xrightarrow{\eta RL} RLRL \xrightarrow{R\varepsilon L} RL, \\ \gamma &:= L(\alpha\beta(I_R)) : LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR, \\ \underline{\gamma} &:= \beta\alpha(L(I_R)) : LR \xrightarrow{L\eta R} LRLR \xrightarrow{\varepsilon LR} LR, \end{aligned}$$

which have the properties

$$\begin{aligned} R\varepsilon L \cdot RL\vartheta &= \vartheta \cdot R\varepsilon L, & R\varepsilon L \cdot \underline{\vartheta}RL &= \underline{\vartheta} \cdot R\varepsilon L, & \underline{\vartheta} \cdot \vartheta &= \vartheta \cdot \underline{\vartheta}; \\ LR\gamma \cdot L\eta R &= L\eta R \cdot \gamma, & \underline{\gamma}LR \cdot L\eta R &= L\eta R \cdot \underline{\gamma}, & \underline{\gamma} \cdot \gamma &= \gamma \cdot \underline{\gamma}. \end{aligned}$$

**2.5. Definitions.** Let  $(L, R, \alpha, \beta)$  be a pairing (see 2.2). We call

$$\begin{aligned} \alpha \text{ regular} &\quad \text{if } \alpha \cdot \beta \cdot \alpha = \alpha; \\ \alpha \text{ symmetric} &\quad \text{if } \vartheta = \underline{\vartheta}; \\ \beta \text{ regular} &\quad \text{if } \beta \cdot \alpha \cdot \beta = \beta; \\ \beta \text{ symmetric} &\quad \text{if } \gamma = \underline{\gamma}; \\ (L, R, \alpha, \beta) \text{ regular} &\quad \text{if } \alpha = \alpha \cdot \beta \cdot \alpha \text{ and } \beta = \beta \cdot \alpha \cdot \beta. \end{aligned}$$

The following properties are easy to verify:

- (i) If  $\alpha$  is regular, then  $\beta \cdot \alpha(I_L)$ ,  $\vartheta$  and  $\underline{\vartheta}$  are idempotent and  $\vartheta \cdot \eta = \eta = \underline{\vartheta} \cdot \eta$ ; furthermore, for  $\beta' := \beta \cdot \alpha \cdot \beta$ ,  $(L, R, \alpha, \beta')$  is a regular pairing.
- (ii) If  $\beta$  is regular, then  $\alpha \cdot \beta(I_R)$ ,  $\gamma$  and  $\underline{\gamma}$  are idempotent and  $\varepsilon \cdot \gamma = \varepsilon = \varepsilon \cdot \underline{\gamma}$ ; furthermore, for  $\alpha' := \alpha \cdot \beta \cdot \alpha$ ,  $(L, R, \alpha', \beta)$  is a regular pairing.

Any pairing  $(L, R, \alpha, \beta)$  with  $\beta \cdot \alpha = I$  or  $\alpha \cdot \beta = I$  is regular. The second condition defines the *semiadjoint functors* in Medvedev [16].

With manipulations known from ring theory one can show how pairings with regular components can be related with adjunctions provided idempotents split.

**2.6. Related adjunctions.** Let  $(L, R, \alpha, \beta)$  be a pairing (with quasi-unit  $\eta$ , quasi-counit  $\varepsilon$ ) and assume  $\alpha$  to be regular.

If the idempotent  $h := \beta \cdot \alpha(I_L) : L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L$  splits, that is, there are a functor  $\underline{L} : \mathbb{A} \rightarrow \mathbb{B}$  and natural transformations

$$p : L \rightarrow \underline{L}, \quad i : \underline{L} \rightarrow L \quad \text{with} \quad i \cdot p = h \quad \text{and} \quad p \cdot i = I_{\underline{L}},$$

then the natural transformations

$$\underline{\eta} : I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{Rp} R\underline{L}, \quad \underline{\varepsilon} : \underline{L}R \xrightarrow{iR} LR \xrightarrow{\varepsilon} I_{\mathbb{B}},$$

as quasi-unit and quasi-counit, define a pairing  $(\underline{L}, R, \underline{\alpha}, \underline{\beta})$  with  $\underline{\beta} \cdot \underline{\alpha} = I$ .

If  $\alpha \cdot \beta = I$ , then  $(\widehat{L}, R, \underline{\alpha}, \underline{\beta})$  is an adjunction.

In case the natural transformation  $\beta$  is regular, similar constructions apply if we assume that the idempotent  $\alpha \cdot \beta(I_R) : R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R$  splits.

The properties of the  $(RL, R\varepsilon R\eta)$  and  $(LR, L\eta R, \varepsilon)$  mentioned in 2.2 motivate the definitions in the next section.

### 3. MONADS AND MODULES

**3.1.  $q$ -unital monads and their modules.** We call  $(F, \mu)$  a *functor with product* (or *non-unital monad*) provided  $F : \mathbb{A} \rightarrow \mathbb{A}$  is an endofunctor on a category  $\mathbb{A}$  and  $\mu : FF \rightarrow F$  is a natural transformation satisfying the associativity condition  $\mu \cdot F\mu = \mu \cdot \mu F$ .

For  $(F, \mu)$ , a (*non-unital*)  $F$ -*module* is defined as an object  $A \in \mathbb{A}$  with a morphism  $\varrho : F(A) \rightarrow A$  in  $\mathbb{A}$  satisfying  $\varrho \cdot F\varrho = \varrho \cdot \mu_A$ .

*Morphisms* between  $F$ -modules  $(A, \varrho)$ ,  $(A', \varrho')$  are morphisms  $f : A \rightarrow A'$  in  $\mathbb{A}$  with  $\varrho' \cdot F(f) = f \cdot \varrho$ . The set of all these is denoted by  $\text{Mor}_F(A, A')$ . With these morphisms, (non-unital)  $F$ -modules form a category which we denote by  $\underline{\mathbb{A}}_F$ .

By the associativity condition on  $\mu$ , for every  $A \in \mathbb{A}$ ,  $(F(A), \mu_A)$  is an  $F$ -module and this leads to the free functor and the forgetful functor,

$$\phi_F : \mathbb{A} \rightarrow \underline{\mathbb{A}}_F, \quad A \mapsto (F(A), \mu_A), \quad U_F : \underline{\mathbb{A}}_F \rightarrow \mathbb{A}, \quad (A, \varrho) \mapsto A.$$

A triple  $(F, \mu, \eta)$  is said to be a  $q$ -*unital monad* on  $\mathbb{A}$  provided  $(F, \mu)$  is a functor with product and  $\eta : I_{\mathbb{A}} \rightarrow F$  is any natural transformation, called a *quasi-unit* (no additional properties are required). One always can define natural transformations

$$\vartheta : F \xrightarrow{F\eta} FF \xrightarrow{\mu} F, \quad \underline{\vartheta} : F \xrightarrow{\eta F} FF \xrightarrow{\mu} F.$$

Note that for any  $A \in \mathbb{A}$ ,  $\vartheta_A$  is in  $\mathbb{A}_F$  and  $\underline{\vartheta}_A$  is not necessarily so.

Given  $q$ -unital monads  $(F, \mu, \eta)$ ,  $(F', \mu', \eta')$  on  $\mathbb{A}$ , a natural transformation  $h : F \rightarrow F'$  is called a *morphism of  $q$ -unital monads* if

$$\mu' \cdot hh = h \cdot \mu \quad \text{and} \quad \eta' = h \cdot \eta.$$

The existence of a quasi-unit allows the following generalisation of the Eilenberg-Moore construction for (unital) monads.

**3.2.  $q$ -unital monads and pairings.** For a  $q$ -unital monad  $(F, \mu, \eta)$  we obtain a pairing  $(\phi_F, U_F, \alpha_F, \beta_F)$  with the maps, for  $A \in \mathbb{A}$ ,  $(B, \varrho) \in \underline{\mathbb{A}}_F$ ,

$$\begin{aligned} \alpha_F : \text{Mor}_F(\phi_F(A), B) &\rightarrow \text{Mor}_{\mathbb{A}}(A, U_F(B)), \quad f \mapsto f \cdot \eta_A, \\ \beta_F : \text{Mor}_{\mathbb{A}}(A, U_F(B)) &\rightarrow \text{Mor}_F(\phi_F(A), B), \quad g \mapsto \varrho \cdot F(g). \end{aligned}$$

The quasi-unit  $\eta$  is called *regular* if  $\alpha_F$  is regular, that is,

$$I_{\mathbb{A}} \xrightarrow{\eta} F = I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{F\eta} FF \xrightarrow{\mu} F,$$

and we say  $\eta$  is *symmetric* if  $\alpha_F$  is so, that is,  $\vartheta = \underline{\vartheta}$ .

An  $F$ -module  $\varrho : F(A) \rightarrow A$  in  $\underline{\mathbb{A}}_F$  is said to be *compatible* if  $\beta_F \alpha_F(\varrho) = \varrho$ , that is

$$F(A) \xrightarrow{\varrho} A = F(A) \xrightarrow{F\eta_A} FF(A) \xrightarrow{\mu_A} F(A) \xrightarrow{\varrho} A.$$

In particular, the natural transformation  $\mu : FF \rightarrow F$  is compatible if

$$FF \xrightarrow{\mu} F = FF \xrightarrow{F\eta F} FFF \xrightarrow{\mu F} FF \xrightarrow{\mu} F.$$

It is easy to see that this implies

$$FF \xrightarrow{\vartheta\vartheta} FF \xrightarrow{\mu} F = FF \xrightarrow{\mu} F.$$

Let  $\underline{\mathbb{A}}_F$  denote the full subcategory of  $\underline{\mathbb{A}}_F$  made up by the compatible  $F$ -modules. If  $\mu$  is compatible, the image of the free functor  $\phi_F$  lies in  $\underline{\mathbb{A}}_F$  and (by restriction or corestriction) we get the functor pair (keeping the notation for the functors)

$$\phi_F : \mathbb{A} \rightarrow \underline{\mathbb{A}}_F, \quad U_F : \underline{\mathbb{A}}_F \rightarrow \mathbb{A},$$

and a pairing  $(\phi_F, U_F, \alpha_F, \beta_F)$  between  $\mathbb{A}$  and  $\underline{\mathbb{A}}_F$ .

Since for  $(A, \varrho)$  in  $\underline{\mathbb{A}}_F$ ,  $\beta_F(I_{U_F(A)}) = \varrho$ , the compatibility condition on  $\varrho$  implies that  $\beta \cdot \alpha \cdot \beta(\varrho) = \beta(\varrho)$ , i.e.,  $\underline{\beta}$  is regular in  $(\phi_F, U_F, \alpha_F, \beta_F)$  when restricted to  $\underline{\mathbb{A}}_F$ .

**3.3. Definition.** A  $q$ -unital monad  $(F, \eta, \mu)$  is called

- r-unital* if  $\eta$  is regular and  $\mu$  is compatible;
- weak monad* if  $(F, \eta, \mu)$  is *r-unital* and  $\eta$  is symmetric.

Summarising the observations from 3.2 we have:

**3.4. Proposition.** Let  $(F, \mu, \eta)$  be a  $q$ -unital monad.

- (1) The following are equivalent:
  - (a)  $(F, \mu, \eta)$  is an *r-unital monad*;
  - (b)  $(\phi_F, U_F, \alpha_F, \beta_F)$  is a regular pairing of functors between  $\mathbb{A}$  and  $\underline{\mathbb{A}}_F$ .
- (2) The following are equivalent:
  - (a)  $(F, \mu, \eta)$  is weak monad;
  - (b)  $(\phi_F, U_F, \alpha_F, \beta_F)$  is a regular pairing between  $\mathbb{A}$  and  $\underline{\mathbb{A}}_F$  with  $\alpha_F$  symmetric.

A quasi-unit  $\eta$  that is regular and symmetric is named *pre-unit* in the literature (e.g. [11, Definition 2.3]); for the notion of a weak monad (also called *demimonad*) see e.g. [3], [4]. In case  $\eta$  is a unit,  $q$ -unital monads, *r-unital monads* and weak monads all are (unital) monads. In (non-unital) algebras over commutative rings, *r-unital monads* are obtained from idempotents while weak monads correspond to central idempotents (see 3.7).

**3.5. Properties of weak monads.** Let  $(F, \mu, \eta)$  be a weak monad.

- (i)  $\vartheta : F \rightarrow F$  is a morphism of  $q$ -unital monads;
- (ii) for any  $(A, \varphi) \in \underline{\mathbb{A}}_F$ ,

$$F(A) \xrightarrow{\varphi} A = F(A) \xrightarrow{\varphi} A \xrightarrow{\eta_A} F(A) \xrightarrow{\varphi} A$$

and  $A \xrightarrow{\eta_A} F(A) \xrightarrow{\varphi} A$  is an idempotent  $F$ -morphism.

In a  $q$ -unital monad  $(F, \mu, \eta)$ , if  $\eta$  is regular, a compatible multiplication for  $F$  can be found. More precisely one can easily show:

**3.6. Proposition.** Let  $(F, \mu, \eta)$  be a  $q$ -unital monad.

- (1) If  $\eta$  is regular, then, for  $\tilde{\mu} := \mu \cdot F\mu \cdot \mu F\eta F : FF \rightarrow F$ ,  $(F, \tilde{\mu}, \eta)$  is an *r-unital monad*.
- (2) If  $\mu$  is compatible, then, for  $\tilde{\eta} := \mu \cdot F\eta \cdot \eta : I_A \rightarrow F$ ,  $(F, \mu, \tilde{\eta})$  is an *r-unital monad*.
- (3) If  $(F, \mu, \eta)$  is an *r-unital monad*, then for

$$\hat{\mu} : FF \xrightarrow{\eta^{FF}\eta} FFFF \xrightarrow{\mu^{FF}} FFF \xrightarrow{\mu^F} FF \xrightarrow{\mu} F,$$

$(F, \hat{\mu}, \eta)$  is a weak monad.

As a special case, we consider  $q$ -unital monads on the category  ${}_R\mathbb{M}$  of modules over a commutative ring  $R$  with unit. In the terminology used here this comes out as follows.

**3.7. Non-unital algebras.** A  $q$ -unital  $R$ -algebra  $(A, m, u)$  is a non-unital  $R$ -algebra  $(A, m)$  with some  $R$ -linear map  $u : R \rightarrow A$ . Put  $e := u(1_R) \in A$ . Then:

- (1)  $u$  is regular if and only if  $e$  is an idempotent in  $A$ .
- (2)  $u$  is regular and symmetric if and only if  $e$  is a central idempotent (then  $Ae$  is a unital  $R$ -subalgebra of  $A$ ).
- (3)  $\mu$  is compatible if and only if  $ab = aeb$  for all  $a, b \in A$ .
- (4) If  $u$  is regular, then  $\tilde{m}(a \otimes b) := aeb$ , for  $a, b \in A$ , defines an  $r$ -unital algebra  $(A, \tilde{m}, u)$  ( $\tilde{m}$  and  $u$  are regular).
- (5) If  $u$  is regular, then  $\hat{m}(a \otimes b) := eaebe$ , for  $a, b \in A$ , defines an  $r$ -unital algebra  $(A, \hat{m}, u)$  with  $u$  symmetric.

Clearly, the  $q$ -unital algebras  $(A, m, u)$  over  $R$  correspond to the  $q$ -unital monads given by  $(A \otimes_R -, m \otimes -, u \otimes -)$  on  ${}_R\mathbb{M}$ .

For an  $A$ -module  $\varrho : A \otimes M \rightarrow M$ , writing as usual  $\varrho(a \otimes m) = am$ , the compatibility condition comes out as  $am = aem$  for all  $a \in A, m \in M$ .

**3.8. Monads acting on functors.** Let  $T : \mathbb{A} \rightarrow \mathbb{B}$  be a functor and  $(G, \mu', \eta')$  a  $q$ -unital monad on  $\mathbb{B}$ . We call  $T$  a left  $G$ -module if there exists a natural transformation  $\varrho : GT \rightarrow T$  such that

$$GGT \xrightarrow{G\varrho} GT \xrightarrow{\varrho} T = GGT \xrightarrow{\mu'T} GT \xrightarrow{\varrho} T,$$

and we call it a *compatible  $G$ -module* if in addition

$$GT \xrightarrow{\varrho} T = GT \xrightarrow{G\eta'} GGT \xrightarrow{\mu'T} GT \xrightarrow{\varrho} T.$$

**3.9. Proposition.** Let  $T : \mathbb{A} \rightarrow \mathbb{B}$  be a functor and  $(G, \mu', \eta')$  a weak monad on  $\mathbb{B}$ . Then the following are equivalent:

- (a) there is a functor  $\overline{T} : \mathbb{A} \rightarrow \mathbb{B}_G$  with  $T = U_G \overline{T}$ ;
- (b)  $T$  is a compatible  $G$ -module.

**Proof.** (b) $\Rightarrow$ (a) Given  $T$  as a compatible  $G$ -module with  $\varrho : GT \rightarrow T$ , a functor with the required properties is

$$\overline{T} : \mathbb{A} \rightarrow \mathbb{B}_G, \quad A \mapsto (T(A), \varrho_A : GT(A) \rightarrow T(A)).$$

(a) $\Rightarrow$ (b) For any  $A \in \mathbb{A}$ , there are morphisms  $\rho_A : GT(A) \rightarrow T(A)$  and we claim that these define a natural transformation  $\rho : GT \rightarrow T$ . For this we have to show that, for any morphism  $f : A \rightarrow \hat{A}$ , the middle rectangle is commutative in the diagram

$$\begin{array}{ccc} GGT(A) & \xrightarrow{\mu'_T(A)} & GT(A) \\ \uparrow G\eta'_T A & & \swarrow \rho_A \\ GT(A) & \xrightarrow{\rho_A} & T(A) \\ \downarrow GT(f) & & \downarrow T(f) \\ GT(\hat{A}) & \xrightarrow{\rho_{\hat{A}}} & T(\hat{A}) \\ \downarrow G\eta'_T \hat{A} & & \swarrow \rho_{\hat{A}} \\ GGT(\hat{A}) & \xrightarrow{\mu'_{T(\hat{A})}} & GT(\hat{A}). \end{array}$$

The top and bottom diagrams are commutative by compatibility of the  $G$ -modules, the right trapezium is commutative since  $T(f)$  is a  $G$ -morphism, and the outer paths commute by symmetry of  $\eta'$ . Thus the inner diagram is commutative showing naturality of  $\rho$ .  $\square$

## 4. COMONADS AND COMODULES

In this section we sketch the transfer of the constructions for monads to comonads.

**4.1.  $q$ -counital comonads and their comodules.** A *functor with coproduct* (or *non-counital comonad*) is a pair  $(G, \delta)$  where  $G : \mathbb{A} \rightarrow \mathbb{A}$  is an endofunctor and  $\delta : G \rightarrow GG$  is a natural transformation subject to the coassociativity condition  $G\delta \cdot \delta = \delta G \cdot \delta$ .

For  $(G, \delta)$ , a (*non-counital*)  $G$ -comodule is defined as an object  $A \in \mathbb{A}$  with a morphism  $v : A \rightarrow G(A)$  in  $\mathbb{A}$  such that  $Gv \cdot v = \delta_A \cdot v$ .

*Morphisms* between  $G$ -comodules  $(A, v)$ ,  $(A', v')$  are morphisms  $g : A \rightarrow A'$  in  $\mathbb{A}$  satisfying  $v' \cdot g = G(g) \cdot v$ , and the set of all these is denoted by  $\text{Mor}^G(A, A')$ . With these morphisms, (non-counital)  $G$ -comodules form a category which we denote by  $\underline{\mathbb{A}}^G$ . For this there are the obvious free and forgetful functors

$$\phi^G : \mathbb{A} \rightarrow \underline{\mathbb{A}}^G, \quad U^G : \underline{\mathbb{A}}^G \rightarrow \mathbb{A}.$$

A triple  $(G, \delta, \varepsilon)$  is said to be a  *$q$ -counital comonad* provided  $(G, \delta)$  is a functor with coproduct and  $\varepsilon : G \rightarrow I_{\mathbb{A}}$  is any natural transformation, called a *quasi-counit*. One can always define natural transformations

$$\gamma : G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G, \quad \underline{\gamma} : G \xrightarrow{\delta} GG \xrightarrow{\varepsilon G} G.$$

Morphisms of  $q$ -counital comonads are defined in an obvious way (dual to 3.1).

**4.2.  $q$ -counital comonads and pairings.** For  $(G, \delta, \varepsilon)$ , the functors  $\phi^G$  and  $U^G$  allow for a pairing  $(U^G, \phi^G, \alpha^G, \beta^G)$  where, for  $A \in \mathbb{A}$  and  $(B, v) \in \underline{\mathbb{A}}^G$ ,

$$\begin{aligned} \alpha^G : \text{Mor}_{\mathbb{A}}(U^G(B), A) &\rightarrow \text{Mor}^G(B, \phi^G(A)), & f &\mapsto G(f) \cdot v, \\ \beta^G : \text{Mor}^G(B, \phi^G(A)) &\rightarrow \text{Mor}_{\mathbb{A}}(U^G(B), A), & g &\mapsto \varepsilon_A \cdot g. \end{aligned}$$

The quasi-counit  $\varepsilon$  is called *regular* if  $\beta^G$  is regular, that is,

$$G \xrightarrow{\varepsilon} I_{\mathbb{A}} = G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G \xrightarrow{\varepsilon} I_{\mathbb{A}},$$

and we say  $\eta$  is *symmetric* provided  $\phi^G$  is so, that is  $\gamma = \underline{\gamma}$ .

A (non-counital)  $G$ -comodule  $(B, v)$  is said to be *compatible* provided  $\alpha^G \beta^G(v) = v$ , that is

$$B \xrightarrow{v} G(B) = B \xrightarrow{v} G(B) \xrightarrow{\delta_B} GG(B) \xrightarrow{G\varepsilon_B} G(B).$$

In particular,  $\delta$  is compatible if

$$G \xrightarrow{\delta} GG = G \xrightarrow{\delta} GG \xrightarrow{\delta G} GGG \xrightarrow{G\varepsilon G} GG.$$

This obviously implies

$$G \xrightarrow{\delta} GG = G \xrightarrow{\delta} GG \xrightarrow{\gamma \gamma} GG.$$

By  $\underline{\mathbb{A}}^G$  we denote the full subcategory of  $\underline{\mathbb{A}}^G$  whose objects are compatible  $G$ -comodules.

If  $\delta$  is compatible, the image of the free functor  $\phi^G$  lies in  $\underline{\mathbb{A}}^G$  and (by restriction and corestriction) we obtain the functor pairing (keeping the notation for the functors)

$$\phi^G : \mathbb{A} \rightarrow \underline{\mathbb{A}}^G, \quad U^G : \underline{\mathbb{A}}^G \rightarrow \mathbb{A},$$

leading to a pairing  $(U^G, \phi^G, \alpha^G, \beta^G)$  between  $\mathbb{A}$  and  $\underline{\mathbb{A}}^G$ .

Since for  $(B, v)$  in  $\underline{\mathbb{A}}^G$ ,  $\alpha^G(I_{U^G(B)}) = v$ , the compatibility condition on  $v$  implies that  $\alpha^G \cdot \beta^G \cdot \alpha^G(v) = \alpha^G(v)$ , i.e.,  $\alpha$  is regular in  $(U^G, \phi^G, \alpha^G, \beta^G)$  when restricted to  $\underline{\mathbb{A}}^G$ .

**4.3. Definition.** A  $q$ -counital comonad  $(G, \delta, \varepsilon)$  is called

- $r$ -counital* if  $\varepsilon$  is regular and  $\delta$  is compatible;
- weak comonad* if it is  $r$ -counital and  $\varepsilon$  is symmetric.

From the constructions above we obtain:

**4.4. Proposition.** *Let  $(G, \delta, \varepsilon)$  be a  $q$ -counital comonad.*

- (1) *The following are equivalent:*
  - (a)  *$(G, \delta, \varepsilon)$  is an  $r$ -counital comonad;*
  - (b)  *$(U^G, \phi^G, \alpha^G, \beta^G)$  is a regular pairing of functors between  $\mathbb{A}$  and  $\underline{\mathbb{A}}^G$ .*
- (2) *The following are equivalent:*
  - (a)  *$(G, \delta, \varepsilon)$  is weak comonad;*
  - (b)  *$(U^G, \phi^G, \alpha^G, \beta^G)$  is a regular pairing of functors between  $\mathbb{A}$  and  $\underline{\mathbb{A}}^G$  with  $\beta^G$  symmetric.*

Similar to the situation for modules, for any (counital) comonad  $(G, \delta, \varepsilon)$ , all non-counital  $G$ -comodules are compatible (i.e.,  $\underline{\mathbb{A}}^G = \underline{\mathbb{A}}^G$ ).

**4.5. Properties of weak comonads.** *Let  $(G, \delta, \varepsilon)$  be a weak comonad.*

- (i)  *$\gamma : G \rightarrow G$  is an idempotent morphism of  $q$ -counital comonads;*
- (ii) *for any  $(B, v) \in \underline{\mathbb{A}}^G$ ,*

$$B \xrightarrow{v} G(B) = B \xrightarrow{v} G(B) \xrightarrow{\varepsilon_B} B \xrightarrow{v} G(B)$$

*and  $B \xrightarrow{v} G(B) \xrightarrow{\varepsilon_B} B$  is an idempotent  $G$ -morphism.*

Properties of pairings can improved in the following sense.

**4.6. Proposition.** *Let  $(G, \delta, \varepsilon)$  be a  $q$ -counital comonad.*

- (1) *If  $\varepsilon$  is regular, then, for  $\tilde{\delta} : G \xrightarrow{\delta} GG \xrightarrow{G\delta} GGG \xrightarrow{G\varepsilon} GG$ ,  $(G, \tilde{\delta}, \varepsilon)$  is an  $r$ -counital comonad.*
- (2) *If  $\delta$  is compatible, then, for  $\tilde{\varepsilon} : G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G \xrightarrow{\varepsilon} I_{\mathbb{A}}$ ,  $(G, \delta, \tilde{\varepsilon})$  is an  $r$ -counital comonad.*
- (3) *If  $(G, \delta, \varepsilon)$  is a regular quasi-comonad, then, for*

$$\hat{\delta} : G \xrightarrow{\delta} GG \xrightarrow{G\delta} GGG \xrightarrow{GG\delta} GGGG \xrightarrow{\varepsilon GG\varepsilon} GG,$$

*$(G, \hat{\delta}, \varepsilon)$  is a weak comonad.*

As a special case, consider non-counital comonads on the category  ${}_R\mathbb{M}$  of modules over a commutative ring  $R$  with unit. In our terminology this comes out as follows.

**4.7. Non-counital coalgebras.** A  $q$ -counital coalgebra  $(C, \Delta, \varepsilon)$  is a non-counital  $R$ -coalgebra  $(C, \Delta)$  with some  $R$ -linear map  $\varepsilon : C \rightarrow R$ . Writing  $\Delta(c) = \sum c_1 \otimes c_2$  for  $c \in C$ , we have:

- (1)  $\varepsilon$  is regular if and only if for any  $c \in C$ ,  $\varepsilon(c) = \sum \varepsilon(c_1)\varepsilon(c_2)$ .
- (2)  $\varepsilon$  is symmetric if and only if  $\sum c_1\varepsilon(c_2) = \sum \varepsilon(c_1)c_2$ .
- (3)  $\Delta$  is compatible if and only if  $\Delta(c) = \sum c_1 \otimes c_2\varepsilon(c_3)$ .
- (4) If  $\varepsilon$  is regular, then  $\tilde{\Delta}(c) := \sum c_1 \otimes \varepsilon(c_2)c_3$  defines an  $r$ -counital coalgebra  $(C, \tilde{\Delta}, \varepsilon)$ .
- (5) If  $(C, \Delta, \varepsilon)$  is an  $r$ -counital comonad, then  $\hat{\Delta}(c) := \sum \varepsilon(c_1)c_2 \otimes c_3\varepsilon(c_4)$  defines an  $r$ -counital coalgebra  $(C, \hat{\Delta}, \varepsilon)$  with  $\varepsilon$  symmetric.

Clearly, the  $q$ -counital coalgebras  $(C, \Delta, \varepsilon)$  over  $R$  correspond to the  $q$ -counital comonads given by  $(C \otimes_R -, \Delta \otimes -, \varepsilon \otimes -)$  on  ${}_R\mathbb{M}$ . From this the compatibility conditions for  $C$ -comodules are derived (see 4.2).

**4.8. Weak corings and pre- $A$ -corings.** Let  $A$  be a ring with unit  $1_A$  and  $\mathcal{C}$  a non-unital  $(A, A)$ -bimodule which is unital as right  $A$ -module. Assume there are  $(A, A)$ -bilinear maps

$$\underline{\Delta} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}, \quad \underline{\varepsilon} : \mathcal{C} \rightarrow A,$$

where  $\underline{\Delta}$  is coassociative.

$(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$  is called a *right unital weak  $A$ -coring* in [21], provided for all  $c \in \mathcal{C}$ ,

$$(\underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot \underline{\Delta}(c) = 1_A \cdot c = (I_{\mathcal{C}} \otimes \underline{\varepsilon}) \cdot \underline{\Delta}(c),$$

which reads in (obvious) Sweedler notation as  $\sum \underline{\varepsilon}(c_1)c_2 = 1_A \cdot c = \sum c_1\underline{\varepsilon}(c_2)$ .

From the equations

$$\begin{aligned} (I_{\mathcal{C}} \otimes \underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot (I_{\mathcal{C}} \otimes \underline{\Delta}) \cdot \underline{\Delta}(c) &= \sum c_1 \otimes 1_A \cdot c_2 = \sum c_1 \otimes c_2 = \underline{\Delta}(c), \\ (I_{\mathcal{C}} \otimes \underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot (\underline{\Delta} \otimes I_{\mathcal{C}}) \cdot \underline{\Delta}(c) &= \sum 1_A \cdot c_1 \otimes c_2 = 1_A \cdot \underline{\Delta}(c), \end{aligned}$$

it follows by coassociativity that  $1_A \cdot \underline{\Delta}(c) = \underline{\Delta}(c)$ . Summarising we see that, in this case,  $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$  induces a weak comonad on the category  ${}_A\mathbb{M}$  of left non-unital  $A$ -modules ( $= {}_A\mathbb{M}$  since  $A$  has a unit).

$(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$  is called an  *$A$ -pre-coring* in [7, Section 6], if

$$(\underline{\varepsilon} \otimes I_{\mathcal{C}}) \cdot \underline{\Delta}(c) = c, \quad (I_{\mathcal{C}} \otimes \underline{\varepsilon}) \cdot \underline{\Delta}(c) = 1_A \cdot c,$$

which reads (in Sweedler notation) as  $c = \sum \underline{\varepsilon}(c_1)c_2$ ,  $1_A \cdot c = \sum c_1\underline{\varepsilon}(c_2)$ .

Similar to the computation above we obtain that  $1_A \cdot \underline{\Delta}(c) = \underline{\Delta}(c)$ . Now  $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$  induces an  $r$ -counital comonad on  ${}_A\mathbb{M}$  but  $\underline{\varepsilon}$  is not symmetric.

Notice that in both cases considered above, restriction and corestriction of  $\underline{\Delta}$  and  $\underline{\varepsilon}$  yield an  *$A$ -coring*  $(\mathcal{A}\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$  (e.g. [21, Proposition 1.3]).

**4.9. Comonads acting on functors.** Let  $T : \mathbb{A} \rightarrow \mathbb{B}$  be a functor and  $(G, \delta, \varepsilon)$  a weak comonad on  $\mathbb{B}$ . We call  $T$  a left (*non-counital*)  $G$ -comodule if there exists a natural transformation  $v : T \rightarrow GT$  such that

$$T \xrightarrow{v} GT \xrightarrow{vG} GGT = T \xrightarrow{vT} GT \xrightarrow{\delta} GGT,$$

and we call it a *compatible  $G$ -comodule* if, in addition,

$$T \xrightarrow{v} GT = T \xrightarrow{v} GT \xrightarrow{\delta} GGT \xrightarrow{G\varepsilon} GT.$$

Dual to Proposition 3.9, given a weak comonad  $(G, \delta, \varepsilon)$  on  $\mathbb{B}$ , a functor  $T : \mathbb{A} \rightarrow \mathbb{B}$  is a compatible  $G$ -comodule if and only if there is a functor  $\overline{T} : \mathbb{A} \rightarrow \mathbb{B}^G$  with  $T = U^G \overline{T}$ .

The motivation for considering generalised monads and comonads came from structures observed while handling full pairings of functors (see end of Section 2). Now we want to reconsider the pairings in view of these constructions.

For any pairing  $(L, R, \alpha, \beta)$  between the categories  $\mathbb{A}$  and  $\mathbb{B}$ ,  $(RL, R\varepsilon L, \eta)$  is a  $q$ -unital monad and  $(LR, L\eta R, \varepsilon)$  is a  $q$ -counital comonad. It is easy to see that

- (i) if  $\beta$  is regular, then for any  $B \in \mathbb{B}$ ,  $R\varepsilon : RLR(B) \rightarrow R(B)$  is a compatible  $RL$ -module.
- (ii) if  $\alpha$  is regular, then for any  $A \in \mathbb{A}$ ,  $L\eta : L(A) \rightarrow LRL(A)$  is a compatible  $LR$ -comodule.

**4.10. Comparison functors.** For a regular pairing  $(L, R, \alpha, \beta)$  between  $\mathbb{A}$  and  $\mathbb{B}$ ,

$(RL, R\varepsilon L, \eta)$  is an  $r$ -unital monad on  $\mathbb{A}$  with a (comparison) functor

$$\widehat{R} : \mathbb{B} \rightarrow \underline{\mathbb{A}}_{RL}, \quad B \mapsto (R(B), R\varepsilon : RLR(B) \rightarrow R(B)),$$

$(LR, L\eta R, \varepsilon)$  is an  $r$ -counital comonad on  $\mathbb{B}$  with a (comparison) functor

$$\widetilde{L} : \mathbb{A} \rightarrow \underline{\mathbb{B}}^{LR}, \quad A \mapsto (L(A), L\eta : L(A) \rightarrow LRL(A)),$$

inducing commutativity of the diagrams

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{L} & \mathbb{B} & \xrightarrow{R} & \mathbb{A} \\ & \searrow \phi_{RL} & \downarrow \widehat{R} & \nearrow U_{RL} & \\ & & \underline{\mathbb{A}}_{RL} & & \end{array}, \quad \begin{array}{ccc} \mathbb{B} & \xrightarrow{R} & \mathbb{A} & \xrightarrow{L} & \mathbb{B} \\ & \searrow \phi^{LR} & \downarrow \widetilde{L} & \nearrow U^{LR} & \\ & & \underline{\mathbb{B}}^{LR} & & \end{array}.$$

It follows from 3.2 that for the  $r$ -unital monad  $(RL, R\varepsilon L, \eta)$ ,  $(\phi_{RL}, U_{RL}, \alpha_{RL}, \beta_{RL})$  is a regular pairing between  $\mathbb{A}$  and  $\underline{\mathbb{A}}_{RL}$ . Similarly, by 4.2, for the  $R$ -counital comonad  $(LR, L\eta R, \varepsilon)$ ,  $(U^{LR}, \phi^{LR}, \alpha^{LR}, \beta^{LR})$  is a regular pairing between  $\mathbb{B}$  and  $\underline{\mathbb{B}}^{LR}$ .

**4.11. Relating  $(L, R)$  with  $(\phi_{RL}, U_{RL})$  and  $(U^{LR}, \phi^{LR})$ .** With the above notions we form the diagram

$$\begin{array}{ccccc} \text{Mor}_{\mathbb{B}}(L(A), B) & \xrightarrow{\alpha} & \text{Mor}_{\mathbb{A}}(A, R(B)) & \xrightarrow{\beta} & \text{Mor}_{\mathbb{B}}(L(A), B) \\ \hat{R}_{-, -} \downarrow & & & & \downarrow \hat{R}_{-, -} \\ \text{Mor}_{RL}(\phi_{RL}(A), R(B)) & \xrightarrow{\alpha_{RL}} & \text{Mor}_{\mathbb{A}}(A, U_{RL}R(B)) & \xrightarrow{\beta_{RL}} & \text{Mor}_{RL}(\phi_{RL}(A), R(B)). \end{array}$$

This diagram is commutative if and only if  $\alpha$  is symmetric (see Definitions 2.5).

Similar constructions apply for  $(L, R)$ ,  $(U^{LR}, \phi^{LR})$  and  $\tilde{L}_{-, -}$ . and  $\beta$  is symmetric if and only if  $\tilde{K}_{-, -} \cdot \alpha \cdot \beta = \alpha^{LR} \cdot \beta^{LR} \cdot \tilde{L}_{-, -}$ .

**4.12. Corollary.** *Consider a pairing  $(L, R, \alpha, \beta)$  (see 2.2).*

- (1) *The following are equivalent:*
  - (a)  $(L, R, \alpha, \beta)$  is a regular pairing;
  - (b)  $(RL, R\varepsilon L, \eta)$  is an  $r$ -unital monad on  $\mathbb{A}$  and  $(LR, L\eta R, \varepsilon)$  is an  $r$ -counital comonad on  $\mathbb{B}$ .
- (2) *The following are equivalent:*
  - (a)  $(L, R, \alpha, \beta)$  is a regular pairing with  $\alpha$  and  $\beta$  symmetric;
  - (b)  $(RL, R\varepsilon L, \eta)$  is a weak monad on  $\mathbb{A}$  and  $(LR, L\eta R, \varepsilon)$  is a weak comonad on  $\mathbb{B}$ .

## 5. ENTWINING MONADS AND COMONADS

**5.1. Lifting of functors to module categories.** Let  $(F, \mu, \eta)$  and  $(L, \mu', \eta')$  be  $r$ -unital monads on the categories  $\mathbb{A}$  and  $\mathbb{B}$ , respectively, and  $\underline{\mathbb{A}}_F, \underline{\mathbb{B}}_L$  the categories of the corresponding compatible modules (see 3.2). Given functors  $T : \mathbb{A} \rightarrow \mathbb{B}$  and  $\bar{T} : \underline{\mathbb{A}}_F \rightarrow \underline{\mathbb{B}}_L$ , we say that  $\bar{T}$  is a *lifting* of  $T$  provided the diagram

$$(5.1) \quad \begin{array}{ccc} \underline{\mathbb{A}}_F & \xrightarrow{\bar{T}} & \underline{\mathbb{B}}_L \\ U_F \downarrow & & \downarrow U_L \\ \mathbb{A} & \xrightarrow{T} & \mathbb{B} \end{array}$$

is commutative, where the  $U$ 's denote the forgetful functors.

**5.2. Proposition.** *With the data given in 5.1, consider the functors  $TF, LT : \mathbb{A} \rightarrow \mathbb{B}$  and a natural transformation  $\lambda : LT \rightarrow TF$ . The non-unital  $F$ -module  $(F, \mu)$  induces an  $L$ -action on  $TF$ ,*

$$\chi : LTF \xrightarrow{\lambda F} TFF \xrightarrow{T\mu} TF.$$

- (1) *If  $(TF, \chi)$  is a (non-unital)  $L$ -module, then we get the commutative diagram*

$$(5.2) \quad \begin{array}{ccccccc} LLT & \xrightarrow{L\lambda} & LTF & \xrightarrow{LT\vartheta} & LTF & \xrightarrow{\lambda F} & TFF \\ \mu'T \downarrow & & & & & & \downarrow T\mu \\ LT & \xrightarrow{\lambda} & TF & \xrightarrow{T\vartheta} & TF & & \end{array}$$

- (2) *If  $(TF, \chi)$  is a compatible  $L$ -module, then (with  $\vartheta' = \mu' \cdot F\eta'$ )*

$$(5.3) \quad LT \xrightarrow{\vartheta'T} LT \xrightarrow{\lambda} TF \xrightarrow{T\vartheta} TF = LT \xrightarrow{\lambda} TF \xrightarrow{T\vartheta} TF.$$

(3) If  $\eta$  is symmetric in  $(F, \mu, \eta)$  and  $(A, \varphi)$  is a compatible  $F$ -module, then

$$(5.4) \quad T\varphi \cdot \lambda_A = T\varphi \cdot \lambda_A \cdot LT\varphi \cdot LT\eta_A.$$

**Proof.** The proof follows essentially as in the monad case replacing the identity on  $F$  at some places by  $\vartheta = \mu \cdot F\eta$  (see 3.1).

To show (3), Proposition 3.5 is needed.  $\square$

**5.3. Proposition.** Let  $(F, \mu, \eta)$  and  $(L, \mu', \eta')$  be  $r$ -unital monads on  $\mathbb{A}$  and  $\mathbb{B}$ , respectively, and  $T : \mathbb{A} \rightarrow \mathbb{B}$  any functor. Then a natural transformation  $\lambda : LT \rightarrow TF$  induces a lifting to the compatible modules,

$$\overline{T} : \underline{\mathbb{A}}_F \rightarrow \underline{\mathbb{B}}_L, \quad (A, \varphi) \mapsto (T(A), T\varphi \cdot \lambda_A : LT(A) \rightarrow T(A)),$$

if and only if the diagram (5.2) is commutative and equation (5.3) holds.

**Proof.** One direction follows from Proposition 5.2, the other one by a slight modification of the proof in the monad case.  $\square$

To show that the lifting property implies the existence of a natural transformation  $\lambda : LT \rightarrow TF$  we need the symmetry of the units, that is, we require the  $r$ -unital monads to be weak monads. Then we can extend Applegate's lifting theorem for monads (and unital modules) (e.g. [13, Lemma 1], [22, 3.3]) to weak monads (and compatible modules).

**5.4. Theorem.** Let  $(F, \mu, \eta)$  and  $(L, \mu', \eta')$  be weak monads on  $\mathbb{A}$  and  $\mathbb{B}$ , respectively. For any functor  $T : \mathbb{A} \rightarrow \mathbb{B}$ , there are bijective correspondences between

- (i) liftings of  $T$  to  $\overline{T} : \underline{\mathbb{A}}_F \rightarrow \underline{\mathbb{B}}_L$ ;
- (ii) compatible  $L$ -module structures  $\varrho$  on  $TU_F : \underline{\mathbb{A}}_F \rightarrow \underline{\mathbb{B}}$ ;
- (iii) natural transformations  $\lambda : LT \rightarrow TF$  with commuting diagrams

$$(5.5) \quad \begin{array}{ccc} LLT & \xrightarrow{L\lambda} & LTF \xrightarrow{\lambda^F} TFF \\ \mu'T \downarrow & & \downarrow T\mu \\ LT & \xrightarrow{\lambda} & TF, \end{array} \quad \begin{array}{ccc} LT & \xrightarrow{\vartheta'T} & LT \\ \lambda \downarrow & \searrow \lambda & \downarrow \lambda \\ TF & \xrightarrow{T\vartheta} & TF. \end{array}$$

**Proof.** (i) $\Leftrightarrow$ (ii) follows by Proposition 3.9.

(ii) $\Rightarrow$ (iii) Given the compatible  $L$ -module structure map  $\varrho$ , put

$$\lambda := \varrho F \cdot LT\eta : LT \xrightarrow{LT\eta} LTF \xrightarrow{\varrho F} TF.$$

Notice that for  $\lambda$  we can take  $T\vartheta \cdot \lambda$  from Proposition 5.2.

(iii) $\Rightarrow$ (i) Given  $\lambda$  with the commutative diagram in (iii), it follows by Propositions 5.3 that  $\varrho_A := T\varphi \cdot \lambda_A$  induces a lifting.  $\square$

**5.5. Lifting of functors to comodules.** Let  $(G, \delta, \varepsilon)$  and  $(H, \delta', \varepsilon')$  be  $r$ -unital comonads on the categories  $\mathbb{A}$  and  $\mathbb{B}$ , respectively, and  $\underline{\mathbb{A}}^G, \underline{\mathbb{B}}^H$  the corresponding categories of the compatible comodules (see 4.2). Given a functor  $T : \mathbb{A} \rightarrow \mathbb{B}$ , a functor  $\widehat{T} : \underline{\mathbb{A}}^G \rightarrow \underline{\mathbb{B}}^H$ , is said to be *a lifting of  $T$*  if the diagram

$$(5.6) \quad \begin{array}{ccc} \underline{\mathbb{A}}^G & \xrightarrow{\widehat{T}} & \underline{\mathbb{B}}^H \\ U^G \downarrow & & \downarrow U^H \\ \mathbb{A} & \xrightarrow{T} & \mathbb{B} \end{array}$$

is commutative where the  $U$ 's denote the forgetful functors.

**5.6. Proposition.** *With the data given in 5.5, consider the functors  $TG, HT : \mathbb{A} \rightarrow \mathbb{B}$  and a natural transformation  $\psi : TG \rightarrow HT$ . The (non-counital)  $G$ -comodule  $(G, \delta)$  induces an  $H$ -coaction on  $TG$ ,*

$$\zeta : TG \xrightarrow{T\delta} TGG \xrightarrow{\psi G} HTG.$$

(1) *If  $(TG, \zeta)$  is a (non-counital)  $H$ -comodule, we get the commutative diagram*

$$(5.7) \quad \begin{array}{ccccc} TG & \xrightarrow{T\gamma} & TG & \xrightarrow{\psi} & HT \\ T\delta \downarrow & & & & \downarrow \delta' T \\ TGG & \xrightarrow{\psi G} & HTG & \xrightarrow{HT\gamma} & HTG \xrightarrow{H\psi} HHT. \end{array}$$

(2) *If  $H(TG, \zeta)$  is a compatible  $H$ -module, then*

$$(5.8) \quad TG \xrightarrow{T\gamma} TG \xrightarrow{\psi} HT \xrightarrow{\gamma' T} HT = TG \xrightarrow{T\gamma} TG \xrightarrow{\psi} HT.$$

(3) *If  $\varepsilon$  is symmetric and  $(A, v)$  is a compatible  $G$ -comodule, then*

$$\psi \cdot Tv = HT\varepsilon \cdot HTv \cdot \psi \cdot Tv.$$

**Proof.** The situation is dual to that of Proposition 5.2.  $\square$

**5.7. Proposition.** *Let  $(G, \delta, \varepsilon)$  and  $(H, \delta', \varepsilon')$  be  $r$ -counital comonads on the categories  $\mathbb{A}$  and  $\mathbb{B}$ , respectively, and  $T : \mathbb{A} \rightarrow \mathbb{B}$  any functor. A natural transformation  $\psi : TG \rightarrow HT$  induces a lifting*

$$\hat{T} : \underline{\mathbb{A}}^G \rightarrow \underline{\mathbb{B}}^H, \quad (A, v) \mapsto (T(A), \psi \cdot Tv : T(A) \rightarrow HT(A)),$$

*if and only if the diagram (5.7) is commutative and equation (5.8) holds.*

**Proof.** The proof is dual to that of Proposition 5.3.  $\square$

Dualising Theorem 5.4, we obtain an extension of Applegate's lifting theorem for comonads (and comodules) (e.g. [22, 3.5]) to weak comonads (and compatible comodules).

**5.8. Theorem.** *Let  $(G, \delta, \varepsilon)$  and  $(H, \delta', \varepsilon')$  be weak comonads on  $\mathbb{A}$  and  $\mathbb{B}$ , respectively. For any functor  $T : \mathbb{A} \rightarrow \mathbb{B}$ , there are bijective correspondences between*

- (i) *liftings of  $T$  to  $\hat{T} : \underline{\mathbb{A}}^G \rightarrow \underline{\mathbb{B}}^H$ ;*
- (ii) *compatible  $H$ -comodule structures  $v : TU^G \rightarrow HTU^G$ ;*
- (iii) *natural transformations  $\psi : TG \rightarrow HT$  with commutative diagrams*

$$\begin{array}{ccc} TG & \xrightarrow{\psi} & HT \\ T\delta \downarrow & & \downarrow \delta' T \\ TGG & \xrightarrow{\psi G} & HTG \xrightarrow{H\psi} HHT, \end{array} \quad \begin{array}{ccc} TG & \xrightarrow{T\gamma} & TG \\ \psi \downarrow & \searrow \psi & \downarrow \psi \\ HT & \xrightarrow{\gamma' T} & HT. \end{array}$$

**Proof.** In view of 5.6 and 5.7, the proof is dual to that of Theorem 5.4. Here we take  $\psi$  as the composition  $\psi \cdot T\gamma$  (with  $\psi$  from 5.6).  $\square$

## 6. LIFTING OF ENDOFUNCTORS TO MODULES AND COMODULES

Given a weak monad  $(F, \mu, \eta)$ , or a weak comonad  $(G, \delta, \varepsilon)$ , and any endofunctor  $T$  on the category  $\mathbb{A}$ , we have learned in the preceding sections when  $T$  can be lifted to an endofunctor of the compatible modules or comodules, respectively. Now, one may also ask if the lifting is again a weak monad or a weak comonad, respectively.

**6.1. Entwining  $r$ -unital monads.** *For weak monads  $(F, \mu, \eta)$  and  $(T, \check{\mu}, \check{\eta})$  on  $\mathbb{A}$  and a natural transformation  $\lambda : FT \rightarrow TF$ , the following are equivalent:*

(a) *defining product and quasi-unit on  $TF$  by*

$$\bar{\mu} : TFFT \xrightarrow{T\lambda F} TTFF \xrightarrow{TT\mu} TTF \xrightarrow{\bar{\mu}F} TF, \quad \bar{\eta} : I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{F\bar{\eta}} FT \xrightarrow{\lambda} TF,$$

*yields a weak monad  $(TF, \bar{\mu}, \bar{\eta})$  on  $\mathbb{A}$ ;*

(b)  *$\lambda$  induces commutativity of the diagrams*

$$(6.1) \quad \begin{array}{ccc} FFT & \xrightarrow{F\lambda} & FTF \xrightarrow{\lambda F} TFF \\ \mu T \downarrow & & \downarrow T\mu \\ FT & \xrightarrow{\lambda} & TF, \end{array} \quad \begin{array}{ccc} FT & \xrightarrow{\vartheta T} & FT \\ \lambda \downarrow & \searrow \lambda & \downarrow \lambda \\ TF & \xrightarrow{T\vartheta} & TF, \end{array}$$

$$(6.2) \quad \begin{array}{ccc} FTT & \xrightarrow{\lambda T} & TFT \xrightarrow{T\lambda} TTF \\ F\bar{\mu} \downarrow & & \downarrow \bar{\mu}F \\ FT & \xrightarrow{\lambda} & TF, \end{array} \quad \begin{array}{ccc} FT & \xrightarrow{F\bar{\vartheta}} & FT \\ \lambda \downarrow & \searrow \lambda & \downarrow \lambda \\ TF & \xrightarrow{\bar{\vartheta}F} & TF; \end{array}$$

(c)  *$\lambda$  induces commutativity of the diagrams in (6.1) and the square in (6.2), and there are natural transformations*

$$\bar{\mu}F : TTF \rightarrow TF \quad \text{and} \quad \lambda \cdot F\bar{\eta} : F \rightarrow TF$$

*where  $\bar{\mu}F$  is a left and right  $F$ -module morphism and  $\lambda \cdot F\bar{\eta}$  is an  $F$ -module morphism.*

*If these conditions hold, we obtain morphisms of  $q$ -unital monads,*

$$\lambda \cdot F\bar{\eta} : F \rightarrow TF \quad \text{and} \quad \lambda \cdot \eta T : T \rightarrow TF.$$

**Proof.** The assertions follow from the general results in Section 5 and some routine computations.  $\square$

**6.2. Weak crossed products.** Given  $(F, \mu, \eta)$  and  $T : \mathbb{A} \rightarrow \mathbb{A}$ , the composition  $TF$  may have a weak monad structure without requiring such a structure on  $T$ . For example, replacing the natural transformations  $\bar{\mu}F$  and  $\lambda \cdot F\bar{\eta}$  in 6.1(c) by some natural transformations

$$\nu : TTF \rightarrow TF, \quad \xi : F \rightarrow TF,$$

similar to 6.1(a), a multiplication and a quasi-unit can be defined on  $TF$ . To make this a weak monad on  $\mathbb{A}$ , special conditions are to be imposed on  $\nu$  and  $\xi$  which can be obtained by routine computations.

Having  $\nu$  and  $\xi$ , one also has natural transformations

$$\bar{\nu} : TT \xrightarrow{TT\eta} TTF \xrightarrow{\nu} TF, \quad \bar{\eta} : I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{\xi} TF,$$

and it is easy to see that  $\bar{\nu}$  leads to the same product on  $TF$  as  $\nu$  does. Thus  $\bar{\nu}$  and  $\bar{\eta}$  may be used to define a weak monad structure on  $TF$  and the conditions required come out as *cocycle* and *twisted conditions*. For more details we refer, e.g., to [1], [11, Section 3].

For a weak comonad  $(G, \delta, \varepsilon)$  and an endofunctor  $T : \mathbb{A} \rightarrow \mathbb{A}$ , we now consider liftings to the category of compatible  $G$ -comodules,  $\hat{T} : \underline{\mathbb{A}}^G \rightarrow \underline{\mathbb{A}}^G$ . The case when  $T$  has a weak comonad structure is dual to 6.1:

**6.3. Entwining weak comonads.** *For weak comonads  $(F, \delta, \varepsilon)$ ,  $(T, \check{\delta}, \check{\varepsilon})$ , and a natural transformation  $\psi : TG \rightarrow GT$ , the following are equivalent:*

(a) *defining a coproduct and quasi-counit on  $TG$  by*

$$\hat{\delta} : TG \xrightarrow{\check{\delta}G} TTG \xrightarrow{TT\check{\delta}} TTGG \xrightarrow{T\psi G} TGTG, \quad \hat{\varepsilon} : TG \xrightarrow{\psi} GT \xrightarrow{G\check{\varepsilon}} G \xrightarrow{\varepsilon} I_{\mathbb{A}},$$

*yields a weak comonad  $(TG, \hat{\delta}, \hat{\varepsilon})$  on  $\mathbb{A}$ ;*

(b)  $\psi$  induces commutativity of the diagrams, where  $\gamma = T\varepsilon \cdot \delta$ ,  $\check{\gamma} = T\check{\varepsilon} \cdot \check{\delta}$ ,

$$(6.3) \quad \begin{array}{ccc} TG & \xrightarrow{\psi} & GT \\ T\delta \downarrow & & \downarrow \delta T \\ TGG & \xrightarrow{\psi G} GTG \xrightarrow{G\psi} & GGT, \end{array} \quad \begin{array}{ccc} TG & \xrightarrow{T\gamma} & TG \\ \psi \downarrow & \searrow \psi & \downarrow \psi \\ GT & \xrightarrow{\gamma T} & GT, \end{array}$$

$$(6.4) \quad \begin{array}{ccc} TG & \xrightarrow{\psi} & GT \\ \delta G \downarrow & & \downarrow G\delta \\ TTG & \xrightarrow{T\psi} TGT \xrightarrow{\psi T} & GTT, \end{array} \quad \begin{array}{ccc} TG & \xrightarrow{\check{\gamma} G} & TG \\ \psi \downarrow & \searrow \psi & \downarrow \psi \\ GT & \xrightarrow{G\check{\gamma}} & GT, \end{array}$$

(c)  $\psi$  induces commutativity of the diagrams (6.3) and the square in (6.4) and we have natural transformations

$$\check{\delta}G : TG \rightarrow TTG, \quad G\check{\varepsilon} \cdot \psi : TG \rightarrow G,$$

where  $\check{\delta}G$  is a left and right  $G$ -comodule morphism and  $G\check{\varepsilon} \cdot \psi$  is a left  $G$ -comodule morphism.

If these conditions hold, we obtain morphisms of  $q$ -unital comonads,

$$G\check{\varepsilon} \cdot \psi : TG \rightarrow G \quad \text{and} \quad \varepsilon T \cdot \psi : TG \rightarrow T.$$

**6.4. Weak crossed coproducts.** In the situation of 6.3, the coproduct on  $TG$  can also be expressed by replacing the natural transformations  $\check{\delta}G$  and  $G\check{\varepsilon} \cdot \psi$  by any natural transformations

$$\nu : TG \rightarrow TTG \quad \text{and} \quad \zeta : TG \rightarrow G,$$

subject to certain conditions to obtain a weak comonad structure on  $TG$ .

Given  $\nu$  and  $\zeta$  as above, one may form

$$\hat{\nu} : TG \xrightarrow{\nu} TTG \xrightarrow{TT\varepsilon} TT, \quad \hat{\zeta} : TG \xrightarrow{\zeta} G \xrightarrow{\varepsilon} I_{\mathbb{A}},$$

and it is easy to see that these induce a weak comonad structure on  $TG$ . This leads to the *weak crossed coproduct* as considered (for coalgebras) in [11] and [12], for example.

## 7. MIXED ENTWININGS AND LIFTINGS

Throughout this section let  $(F, \mu, \eta)$  denote a weak monad and  $(G, \delta, \varepsilon)$  a weak comonad on any category  $\mathbb{A}$ . In this section we investigate the lifting properties to compatible  $F$ -modules and compatible  $G$ -comodules, respectively.

**7.1. Liftings of monads and comonads.** Consider the diagrams

$$\begin{array}{ccc} \underline{\mathbb{A}}_F & \xrightarrow{\overline{G}} & \underline{\mathbb{A}}_F \\ U_F \downarrow & & \downarrow U_F \\ \mathbb{A} & \xrightarrow{G} & \mathbb{A}, \end{array} \quad \begin{array}{ccc} \underline{\mathbb{A}}^G & \xrightarrow{\widehat{F}} & \underline{\mathbb{A}}^G \\ U^G \downarrow & & \downarrow U^G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{A}. \end{array}$$

In both cases the lifting properties are related to a natural transformation

$$\omega : FG \rightarrow GF.$$

The lifting in the left hand case requires commutativity of the diagrams (Proposition 5.3)

$$(7.1) \quad \begin{array}{ccc} FFG & \xrightarrow{F\omega} & FGF \xrightarrow{\omega F} GFF \\ \mu G \downarrow & & \downarrow G\mu \\ FG & \xrightarrow{\omega} & GF, \end{array} \quad \begin{array}{ccc} FG & \xrightarrow{\omega} & GF \\ \vartheta G \downarrow & \searrow \omega & \downarrow G\vartheta \\ FG & \xrightarrow{\omega} & GF, \end{array}$$

whereas the lifting to  $\underline{\mathbb{A}}^G$  needs commutativity of the diagrams (Proposition 5.7)

$$(7.2) \quad \begin{array}{ccc} FG & \xrightarrow{\omega} & GF \\ F\delta \downarrow & & \downarrow \delta F \\ FGG & \xrightarrow{\omega G} & GFG \xrightarrow{G\omega} GGF, \end{array} \quad \begin{array}{ccc} FG & \xrightarrow{\omega} & GF \\ F\gamma \downarrow & \searrow \omega & \downarrow \gamma F \\ FG & \xrightarrow{\omega} & GF. \end{array}$$

To make  $\overline{G}$  a non-counital comonad with coproduct  $\delta$ , the latter has to be an  $F$ -module morphism, in particular,  $\delta F : GF \rightarrow GGF$  has to be an  $F$ -morphism and this follows by commutativity of the rectangle in (7.2) provided the square in (7.1) is commutative.

To make the lifting  $\widehat{F}$  a non-unital monad with multiplication  $\mu$ , the latter has to be a  $G$ -comodule morphism, in particular,  $\mu G : FFG \rightarrow FG$  has to be a  $G$ -module morphism and this follows by commutativity of the rectangle in (7.1) provided the square in (7.2) is commutative.

**7.2. Natural transformations.** The data given in 7.1 allow for natural transformations

$$\begin{aligned} \xi : G &\xrightarrow{\eta G} FG \xrightarrow{\omega} GF \xrightarrow{\varepsilon F} F, \\ \widehat{\kappa} : GF &\xrightarrow{\eta GF} FGF \xrightarrow{\omega F} GFF \xrightarrow{G\mu} GF, \\ \widehat{\tau} : FG &\xrightarrow{F\delta} FGG \xrightarrow{\omega G} GFG \xrightarrow{\varepsilon FG} FG, \end{aligned}$$

with the properties

$$\begin{aligned} G\mu \cdot \widehat{\kappa} F &= \widehat{\kappa} \cdot G\mu, & \widehat{\tau} G \cdot F\delta &= F\delta \cdot \widehat{\tau}, \\ \mu \cdot \xi F &= \varepsilon F \cdot \widehat{\kappa}, & \xi G \cdot \delta &= \widehat{\tau} \cdot \eta G. \end{aligned}$$

- (i) If the rectangle in (7.1) is commutative, then  $\widehat{\kappa}$  is idempotent.
- (ii) If the rectangle in (7.2) is commutative, then  $\widehat{\tau}$  is idempotent.

To make the liftings weak comonads or weak monads, respectively, we have to find pre-units or pre-counits, respectively. In what follows we consider these questions.

**7.3. Lemma.** (Pre-counits for  $\overline{G}$ ) Assume the diagrams in (7.1) to be commutative. Then the following are equivalent:

- (a) for any  $(A, \varphi) \in \underline{\mathbb{A}}_F$ ,  $\varepsilon_A : G(A) \rightarrow A$  is an  $F$ -module morphism;
- (b)  $\varepsilon F : GF \rightarrow F$  is an  $F$ -morphism;
- (c)  $\vartheta = \mu \cdot F\eta$  induces commutativity of the diagram

$$(7.3) \quad \begin{array}{ccc} FG & \xrightarrow{F\varepsilon} & F \\ \omega \downarrow & & \downarrow \vartheta \\ GF & \xrightarrow{\varepsilon F} & F. \end{array}$$

If these conditions are satisfied, then (with  $\gamma = G\varepsilon \cdot \vartheta$ )

$$\mu G \cdot F\widehat{\tau} = \widehat{\tau} \cdot \mu G \quad \text{and} \quad \widehat{\tau} = \vartheta \gamma.$$

**Proof.** This is shown by straightforward verification. □

**7.4. Proposition.** *Assume the diagrams in (7.1), (7.2) and (7.3) to be commutative. Then  $(\overline{G}, \delta, \varepsilon)$  is a weak comonad on  $\underline{\mathbb{A}}_F$ .*

**Proof.** This follows from the preceding observations.  $\square$

Dual to Lemma 7.3 and 7.4 we obtain for the quasi-units for  $\widehat{F}$ :

**7.5. Lemma.** *(Pre-units for  $\widehat{F}$ ) Assume the diagrams in (7.2) to be commutative. Then the following are equivalent:*

- (a) *for any  $(A, v) \in \underline{\mathbb{A}}^G$ ,  $\eta_A : A \rightarrow F(A)$  is a  $G$ -comodule morphism;*
- (b)  *$\eta G : G \rightarrow FG$  is  $G$ -colinear;*
- (c)  *$\gamma = G\varepsilon \cdot \delta$  induces commutativity of the diagram*

$$(7.4) \quad \begin{array}{ccc} G & \xrightarrow{\eta G} & FG \\ \gamma \downarrow & & \downarrow \omega \\ G & \xrightarrow{G\eta} & GF. \end{array}$$

*If these conditions are satisfied, then*

$$G\widehat{\kappa} \cdot \delta F = \delta F \cdot \widehat{\kappa} \quad \text{and} \quad \widehat{\kappa} = \gamma\vartheta.$$

Summing up the above observations yields the

**7.6. Proposition.** *Assume the diagrams in (7.1), (7.2) and (7.4) to be commutative. Then  $(\widehat{F}, \mu, \eta)$  is a weak monad on  $\underline{\mathbb{A}}^G$ .*

One may consider alternative choices for a pre-counit for  $\overline{G}$  or a pre-unit for  $\widehat{F}$ .

**7.7. Lemma.** *Assume the diagrams in (7.1) to be commutative. With the notations from 7.2, the following are equivalent:*

- (a) *for any  $(A, \varphi) \in \underline{\mathbb{A}}_F$ ,  $\overline{\varepsilon}_A : G(A) \xrightarrow{\xi_A} F(A) \xrightarrow{\varphi} A$  is an  $F$ -module morphism;*
- (b)  *$\overline{\varepsilon} F : GF \xrightarrow{\xi F} FF \xrightarrow{\mu} F$  ( $= GF \xrightarrow{\widehat{\kappa}} GF \xrightarrow{\varepsilon F} F$ ) is an  $F$ -morphism;*
- (c) *commutativity of the diagram*

$$(7.5) \quad \begin{array}{ccccc} FFG & \xrightarrow{F\omega} & FGF & \xrightarrow{F\varepsilon F} & FF \\ F\eta G \uparrow & & & & \downarrow \mu \\ FG & \xrightarrow{\omega} & GF & \xrightarrow{\varepsilon F} & F. \end{array}$$

*If these conditions are satisfied, then*

$$\widehat{\tau} = \mu G \cdot F\widehat{\tau} \cdot F\eta G.$$

**Proof.** The proof can be obtained by some diagram constructions.  $\square$

Notice that commutativity of (7.3) implies commutativity of (7.5).

**7.8. Lemma.** *Assume the diagrams in (7.2) to be commutative. Then the following are equivalent:*

- (a) *for any  $(A, v) \in \underline{\mathbb{A}}^G$ ,  $\widehat{\eta} : A \xrightarrow{v} G(A) \xrightarrow{\xi_A} F(A)$  is a  $G$ -comodule morphism;*
- (b)  *$\widehat{\eta} G : G \xrightarrow{\eta G} FG \xrightarrow{\widehat{\tau}} FG$  ( $= G \xrightarrow{\delta} GG \xrightarrow{\xi G} FG$ ) is  $G$ -colinear;*

(c) *commutativity of the diagram*

$$(7.6) \quad \begin{array}{ccccc} G & \xrightarrow{\eta G} & FG & \xrightarrow{\omega} & GF \\ \delta \downarrow & & & & \uparrow G\varepsilon F \\ GG & \xrightarrow{G\eta G} & GFG & \xrightarrow{G\omega} & GGF. \end{array}$$

If these conditions are satisfied, then

$$\widehat{\kappa} = G\varepsilon F \cdot G\widehat{\kappa} \cdot \delta F.$$

**Proof.** The situation is dual to Lemma 7.7.  $\square$

Notice that commutativity of (7.4) implies commutativity of (7.6).

**7.9. Proposition.** *With the data given in 7.1, assume the diagrams in (7.1), (7.2) and (7.5) to be commutative.*

- (1) *If (7.6) is commutative, then  $\bar{\varepsilon}$  from 7.7 is regular for  $\delta$ , and for  $\bar{\delta} : G \rightarrow GG$  with*

$$\bar{\delta}F : GF \xrightarrow{\delta F} GGF \xrightarrow{G\bar{\kappa}} GGF,$$

*$(\bar{G}, \bar{\delta}, \bar{\varepsilon})$  is an  $r$ -counital comonad on  $\underline{\mathbb{A}}_F$ .*

- (2) *If (7.4) is commutative, then  $\bar{\delta}F = \delta F \cdot \widehat{\kappa}$  and  $(\bar{G}, \bar{\delta}, \bar{\varepsilon})$  is a weak comonad on  $\underline{\mathbb{A}}_F$ .*

**Proof.** This can be shown by suitable diagram constructions.  $\square$

**7.10. Proposition.** *With the data given in 7.1, assume the diagrams in (7.1), (7.2), and (7.6) to be commutative.*

- (1) *If (7.5) is commutative, then  $\widehat{\eta}$  in 7.8 is regular for  $\mu$ , and for  $\widehat{\mu} : FF \rightarrow F$  with*

$$\widehat{\mu}G : FFG \xrightarrow{F\widehat{\tau}} FFG \xrightarrow{\mu G} FG,$$

*$(\widehat{F}, \widehat{\mu}, \widehat{\eta})$  is an  $r$ -unital monad on  $\underline{\mathbb{A}}^G$ .*

- (2) *If (7.3) is commutative, then  $\widehat{\mu}G = \widehat{\tau} \cdot \mu G$  and  $(\widehat{F}, \widehat{\mu}, \widehat{\eta})$  is a weak monad on  $\underline{\mathbb{A}}^G$ .*

**Proof.** This is dual to Proposition 7.9.  $\square$

**Acknowledgments.** The author wants to thank Gabriella Böhm, Tomasz Brzeziński and Bachuki Mesablishvili for their interest in a previous version of this paper and for helpful comments on the subject.

## REFERENCES

- [1] Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R., and Rodríguez Raposo, A.B., *Crossed products in weak contexts*, Appl. Categ. Struct. 18(3) (2010), 231-258.
- [2] Beck, J., *Distributive laws*, [in:] *Seminar on Triples and Categorical Homology Theory*, B. Eckmann (ed.), Springer LNM 80 (1969), 119-140.
- [3] Böhm, G., *The weak theory of monads*, Adv. Math. 225(1) (2010), 1-32.
- [4] Böhm, G., Lack, S. and Street, R., *On the 2-category of weak distributive laws*, Commun. Algebra 39(12) (2011), 4567-4583.
- [5] Böhm, G., Lack, S. and Street, R., *Idempotent splittings, colimit completion, and weak aspects of the theory of monads*, J. Pure Appl. Algebra 216 (2012), 385-403.
- [6] Böhm, G., Nill, F. and Szlachányi, K., *Weak Hopf algebras I: Integral theory and  $C^*$ -structure*, J. Algebra 221(2) (1999), 385-438.
- [7] Brzeziński, T., *The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, Alg. Rep. Theory 5 (2002), 389-410.
- [8] Brzeziński, T. and Wisbauer, R., *Corings and Comodules*, London Math. Soc. Lecture Note Series 309, Cambridge University Press (2003).

- [9] Caenepeel, S. and De Groot, E., *Modules over weak entwining structures*, Andruskiewitsch, N. (ed.) et al., New trends in Hopf algebra theory. Proc. Coll. quantum groups and Hopf algebras, La Falda, Argentina, 1999. Providence, Amer. Math. Soc., Contemp. Math. 267 (2000), 31-54.
- [10] Eilenberg, S. and Moore, J.C., *Adjoint functors and triples*, Ill. J. Math. 9 (1965), 381-398.
- [11] Fernández Vilaboa, J.M., González Rodríguez, R. and Rodríguez Raposo, A.B., *Preunits and weak crossed products*, J. Pure Appl. Algebra 213(12) (2009), 2244-2261.
- [12] Fernández Vilaboa, J.M., González Rodríguez, R. and Rodríguez Raposo, A.B., *Weak Crossed Biproducts and Weak Projections*, arXiv:0906.1693 (2009).
- [13] Johnstone, P.T., *Adjoint lifting theorems for categories of modules*, Bull. Lond. Math. Soc. 7 (1975), 294-297.
- [14] Kasch, F. and Mader, A., *Regularity and substructures of Hom*, Frontiers in Mathematics, Birkhäuser Basel (2009)
- [15] Lack, S. and Street, R., *The formal theory of monads II*, J. Pure Appl. Algebra 175(1-3) (2002), 243-265.
- [16] Medvedev, M.Ya., *Semiadjoint functors and Kan extensions*, Sib. Math. J. 15 (1974), 674-676; translation from Sib. Mat. Zh. 15 (1974), 952-956.
- [17] Mesablishvili, B. and Wisbauer, R., *Bimonads and Hopf monads on categories*, J. K-Theory 7(2) (2011), 349-388.
- [18] Mesablishvili, B. and Wisbauer, R., *On Rational Pairings of Functors*, arXiv:1003.3221 (2010), to appear in Appl. Cat. Struct., DOI: 10.1007/s10485-011-9264-1
- [19] Pareigis, B., *Kategorien und Funktoren*, Mathematische Leitfäden, Teubner Verlag, Stuttgart (1969).
- [20] Street, R., *The formal theory of monads*, J. Pure Appl. Algebra 2 (1972), 149-168.
- [21] Wisbauer, R., *Weak corings*, J. Algebra 245(1) (2001), 123-160.
- [22] Wisbauer, R., *Algebras versus coalgebras*, Appl. Categ. Struct. 16(1-2) (2008), 255-295.
- [23] Wisbauer, R., *Lifting theorems for tensor functors on module categories*, J. Algebra Appl. 10(1) (2011), 129-155.

DEPARTMENT OF MATHEMATICS, HEINRICH HEINE UNIVERSITY DÜSSELDORF, GERMANY  
 E-MAIL: WISBAUER@MATH.UNI-DUESSELDORF.DE